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Solutions of the graded classical Yang–Baxter equation and integrable models

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Abstract. Linear superalgebraic equations giving rise to solutions of the graded classical Yang-Baxter equation are developed and solved explicitly. The connection of these equations with the theory of Lie bi-superalgebras is pointed out, and the possibility of using the solutions of the graded classical Yang-Baxter equation to construct integrable supersymmetric models is discussed.

1. Introduction

The Yang-Baxter equations have long been known to embody the underlying symmetries of two-dimensional integrable models [1-3], and recent research has also revealed that they have a profound connection with many branches of mathematical physics and pure mathematics, notably conformal field theories [4], quantum groups [4, 5] and quantum supergroups [6, 7], Lie bi-algebras [8] and knot theory [9].

Let V be a finite-diemensional vector space which may or may not be $\mathbb{Z}_2$-graded; then the quantum Yang-Baxter equation in $\mathrm{End}(V\otimes V\otimes V)$ reads

$$R_{12}(x)R_{13}(xy)R_{23}(y)=R_{23}(y)R_{13}(xy)R_{12}(x) \qquad x, y\in\mathbb{C} \tag{1}$$

where $R_{12}(x)$ etc stem from an $R(x)\in\mathrm{End}(V\otimes V)$. Expressing $R(x)$ as

$$R(x)=\sum_s E^s(x)\otimes D_s(x) \qquad E^s(x), D_s(x)\in\mathrm{End}(V) \qquad \forall s$$

then we have

$$R_{12}(x)=\sum_s E^s(x)\otimes D_s(x)\otimes I$$

$$R_{13}(x)=\sum_s E^s(x)\otimes I\otimes D_s(x)$$

$$R_{23}(x)=\sum_s I\otimes E^s(x)\otimes D_s(x).$$

Note that when V is $\mathbb{Z}_2$-graded, $\otimes$ should be understood as the graded tensor product [10]. When the matrix elements of $R(x)$ are interpreted as Boltzmann weights of two-dimensional lattice models, each non-trivial solution of the Yang-Baxter equation gives rise to an integrable statistical mechanics model.

A systematic classification of the solutions of the nonlinear functional equation (1) is beyond reach at present. However, a systematic method does exist for constructing trigonometric and rational solutions, using the recently introduced techniques [6, 11] of quantum groups and quantum supergroups.

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Just as the classical limit of a quantum mechanical system is recovered when Planck's constant $\hbar$ is sent to zero, the classical Yang–Baxter equation is obtained as the lowest-order non-trivial term in the expansion of equation (1) in powers of a small dimensionless constant η which can be regarded as proportional to $\hbar$. Let us write

$$R(x)=K(x)[I\otimes I+\eta r(x)+O(\eta^2)] \tag{2}$$

where $K(x)$ is a scalar function. Inserting (2) into (1) and considering the lowest-order surviving terms, we obtain the nonlinear functional equation

$$[r_{12}(x), r_{13}(xy)]+[r_{12}(x), r_{23}(y)]+[r_{13}(xy), r_{23}(y)]=0 \tag{3}$$

where the notation $r_{12}(x)$ etc is self-evident. In the literature, this equation is sometimes also written as

$$[r_{12}(u), r_{13}(u+v)]+[r_{12}(u), r_{23}(v)]+[r_{13}(u+v), r_{23}(v)]=0. \tag{3$'$}$$

Obviously (3′) can be transformed into the form (3) upon using the reparametrization $x=e^u$, $y=e^v$.

Equation (3) or (3′) is the classical Yang–Baxter equation, which plays an important role in the study of classical integrable systems [2]. It also arises naturally in the theory of Lie bi-algebras as a consistency condition for a quasitriangular or triangular Lie bi-algebra [8]. For convenience of reference, we will call (3) (and (3′)) the graded classical Yang–Baxter equation when V is a $\mathbb{Z}_2$-graded vector space, and refer to it as the classical Yang–Baxter equation when V is an ordinary vector space.

If V is a g-module of a Lie algebra or a Lie superalgebra g, and thus affords a representation π of g, it is conceivable that (3) admits solutions of the form $r\in\pi(g)\otimes\pi(g)$. In fact *universal* classical r-matrices in $g\otimes g$ can be constructed for (3). For any given representation $\pi(g)$, such an r-matrix then leads to a solution in $\pi(g)\otimes\pi(g)$. In the remainder of this paper, we will regard (3) and (3′) as holding in $g\otimes g\otimes g$, with g being a Lie algebra or Lie superalgebra.

When g is a finite-dimensional simple algebra, the solutions of (3) have been classified by Belavin and Drinfeld [3]. Those results were extended by Leites and Serganova [12] to the graded case with g a simple Lie superalgebra. They gave very general forms for the solutions of (3′) for any such g, but the structure of these general forms is by no means simple. Earlier, some examples of solutions in the graded case were given by Kulish and Sklyanin [13]. Subsequent to these studies, an approach to the quantized Yang–Baxter equation (1) has appeared, for both ungraded and graded cases, based on the theory of quantum groups [5] and quantum supergroups [6, 7] respectively. This approach yields solutions of the nonlinear equation (1) in a very direct way, in effect replacing the nonlinear problem with a linear one.

It might be expected that a correspondingly direct method of attacking the classical equation (3) could be obtained by considering the classical limit of the quantum group (or supergroup) approach to (1). The main purpose of the present work is to show that this is indeed the case. We will study the case when g is a simple basic classical Lie superalgebra [14]. In particular, we will derive some linear algebraic equations by studying the classical limit of the quantum supergroup equations developed in [6]. The solutions of these equations, which automatically satisfy the graded classical Yang–Baxter equation, are constructed explicitly. The possibility of constructing integrable supersymmetric systems from such solutions of the graded classical Yang–Baxter is then discussed.

The present work reveals that there is an intimate relation between the theory of Lie bi-superalgebras and the graded classical Yang–Baxter equation. This of course is not surprising, knowing the natural connection between the classical Yang–Baxter equation and Lie bi-algebras. A detailed discussion of this relation will be reported elsewhere [15].

2. Linearizing the graded classical Yang–Baxter equation

In this section we will examine the classical limit of the quantum supergroup equations developed in [6], in order to obtain two sets of linear superalgebraic equations which determine solutions of the graded classical Yang–Baxter equation.

The quantum supergroup $U_q(g)$ is the quantized universal enveloping algebra of a simple basic classical Lie superalgebra g; it has the structure of a graded Hopf algebra. Let α_i, $i = 1, 2, \ldots, N$, $N = \text{rank of } g$, be a given set of simple roots of g, such that there is only one odd simple root α_s. Denote by $H = \{h_{\alpha_i} \mid i = 1, 2, \ldots, N\}$ the vector space spanned by the Cartan generators h_{α_i} of g, and by $(\, , \,)$ the invariant bilinear form on $H^* = \bigoplus_{i=1}^N \mathbb{C}\alpha_i$. Define the matrix (a_{ij}) by

$$a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \qquad \forall j \quad \text{if } (\alpha_i, \alpha_i) \neq 0$$

and

$$a_{sj} = (\alpha_s, \alpha_j) \qquad \forall j \quad \text{if } (\alpha_s, \alpha_s) = 0$$

where α_s is the unique odd simple root. For a non-zero parameter $q \in \mathbb{C}$, we let

$$q_i = \begin{cases} q^{(\alpha_i, \alpha_i)/2} & (\alpha_i, \alpha_i) \neq 0 \\ q & \text{otherwise} \end{cases}$$

and consider the algebra generated by $\{K_i^{\pm 1} = q^{\pm h_{\alpha_i}}, \hat{e}_{\alpha_i}, \hat{f}_{\alpha_i} \mid i = 1, 2, \ldots, N\}$ subject to the following constraints

$$\begin{aligned}
&[K_i, K_j] = 0 \qquad K_i \hat{e}_{\alpha_j} K_i^{-1} = q_i^{a_{ij}} e_j \qquad K_i \hat{f}_{\alpha_j} K_i^{-1} = q_i^{-a_{ij}} f_{\alpha_j} \\
&[\hat{e}_{\alpha_i}, \hat{f}_{\alpha_j}\} = \delta_{ij}(K_i - K_i^{-1})/(q_i - q_i^{-1}) \\
&\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1 - a_{ij} \\ \nu \end{Bmatrix}_{q_i} \hat{e}_{\alpha_i}^{1-a_{ij}-\nu} \hat{e}_{\alpha_j} \hat{e}_{\alpha_i}^{\nu} = 0 \qquad i \neq j \qquad (\alpha_i, \alpha_i) \neq 0 \\
&\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1 - a_{ij} \\ \nu \end{Bmatrix}_{q_i} \hat{f}_{\alpha_i}^{1-a_{ij}-\nu} \hat{f}_{\alpha_j} \hat{f}_{\alpha_i}^{\nu} = 0 \qquad i \neq j \qquad (\alpha_i, \alpha_i) \neq 0 \\
&\hat{e}_{\alpha_i}^2 = \hat{f}_{\alpha_i}^2 = 0 \qquad \text{if} \quad (\alpha_i, \alpha_i) = 0
\end{aligned}$$

where

$$\begin{bmatrix} 1 - a_{ij} \\ \nu \end{Bmatrix}_{q_i} = \begin{cases} \begin{bmatrix} 1 - a_{ij} \\ \nu \end{bmatrix}_{q_i}^{(-)} & i \neq s \\ (-1)^{(1/2)\nu(\nu-1)} \begin{bmatrix} 1 - a_{ij} \\ \nu \end{bmatrix}_{q_i}^{(+)} & i = s \end{cases}$$

with

$$\begin{bmatrix} m \\ n \end{bmatrix}_q^{(\pm)} = \begin{cases} \dfrac{(q^m \pm q^{-m})(q^{m-1} \pm q^{-m+1}) \ldots (q^{m-n+1} \pm q^{-m+n-1})}{(q \pm q^{-1})(q^2 \pm q^{-2}) \ldots (q^n \pm q^{-n})} & m > n > 0 \\ 1 & n = 0, m. \end{cases}$$

It is a matter of straightforward manipulation to check case by case for all simple basic classical Lie superalgebras that $\Delta: U_q(g) \to U_q(g) \otimes U_q(g)$, with

$$\begin{aligned} \Delta(\hat{e}_{\alpha_i}) &= \hat{e}_{\alpha_i} \otimes q^{-h_{\alpha_i}/2} + q^{h_{\alpha_i}/2} \otimes \hat{e}_{\alpha_i} \\ \Delta(\hat{f}_{\alpha_i}) &= \hat{f}_{\alpha_i} \otimes q^{-h_{\alpha_i}/2} + q^{h_{\alpha_i}/2} \otimes \hat{f}_{\alpha_i} \\ \Delta(q^{h_{\alpha_i}}) &= q^{h_{\alpha_i}} \otimes q^{h_{\alpha_i}} \end{aligned} \tag{4}$$

defines an algebra homomorphism, which we call the co-multiplication on $U_q(g)$. The algebra homomorphism $\varepsilon: U_q(g) \to \mathbb{C}$ defined by

$$\varepsilon(\hat{e}_{\alpha_i}) = \varepsilon(\hat{f}_{\alpha_i}) = 0 \qquad \varepsilon(q^{h_{\alpha_i}}) = \varepsilon(1) = 1$$

determines a co-unit for $U_q(g)$; and an antipode $s: U_q(g) \to U_q(g)$ is defined by

$$\begin{aligned} s(q^{h_{\alpha_i}}) &= q^{-h_{\alpha_i}} \\ s(\hat{e}_{\alpha_i}) &= -q^{-(\alpha_i, \alpha_i)/2} \hat{e}_{\alpha_i} \\ s(\hat{f}_{\alpha_i}) &= -q^{(\alpha_i, \alpha_i)/2} \hat{f}_{\alpha_i} \end{aligned}$$

which we extend to an algebra anti-homomorphism for all of $U_q(g)$, so that

$$s(uv) = (-1)^{[u][v]} s(v) s(u)$$

thus turning $U_q(g)$ into a $\mathbb{Z}_2$-graded Hopf algebra. In the above $[u]$, as usual, denotes the parity of $u \in U_q(g)$, defined for homogeneous elements by

$$[uv] = [u] + [v] \qquad \forall u, v \in U_q(g)$$

$$[\hat{e}_{\alpha_s}] = [\hat{f}_{\alpha_s}] = 1 \qquad [\hat{e}_{\alpha_i}] = [\hat{f}_{\alpha_i}] = 0 \qquad \forall i \neq s \qquad [h_{\alpha_j}] = 0 \qquad \forall j.$$

It is worth noting that in the limit $q \to 1$, $U_q(g)$ reduces to the universal enveloping algebra of g, and the co-multiplication to the familiar diagonal homomorphism $\partial: g \to g \otimes g$ defined by $\partial(a) = a \otimes 1 + 1 \otimes a, \forall a \in g$. We have shown in [6] that when g is a simple basic classical Lie superalgebra, the following equations for $R(x) \in U_q(g)$, R even,

$$\begin{aligned} &R(x)\Delta(u) = \bar{\Delta}(u) R(x) \qquad \forall u \in U_q(g) \\ &R(x)[x\hat{e}_{\alpha_0} \otimes q^{-h_{\alpha_0}/2} + q^{h_{\alpha_0}/2} \otimes \hat{e}_{\alpha_0}] = [x\hat{e}_{\alpha_0} \otimes q^{h_{\alpha_0}/2} + q^{-h_{\alpha_0}/2} \otimes \hat{e}_{\alpha_0}] R(x) \end{aligned} \tag{5}$$

admit at most one non-trivial solution up to scalar multiples, and their solution automatically satisfies the Yang–Baxter equation (1). In (5), it should be understood that $U_q(g)$ is in an irreducible representation π afforded by the finite dimensional irreducible $U_q(g)$-module V, and $R(x) \in \pi(U_q(g)) \otimes \pi(U_q(g))$. The $\hat{e}_{\alpha_0} \in U_q(g)$ is the element of weight α_0, where α_0 is the lowest root of g, satisfying certain defining relations such as

$$[\hat{e}_{\alpha_0}, \hat{f}_{\alpha_i}\} = 0 \qquad i = 1, 2, \ldots, N$$

etc. In the limit $q \to 1$, $\hat{e}_{\alpha_0}$ reduces to e_{α_0}, the lowest root vector of g. The $\bar{\Delta}$ appearing on the right-hand side of the first of equations (5) is the opposite co-product, defined by

$$\bar{\Delta} = T \cdot \Delta \tag{6}$$

with $T: U_q(g) \otimes U_q(g) \to U_q(g) \otimes U_q(g)$ the twisting map defined by

$$T(u \otimes v) = (-1)^{[u][v]} v \otimes u \tag{7}$$

for homogeneous elements $u, v \in U_q(g)$, and extended to all of $U_q(g) \otimes U_q(g)$ by linearity.

In order to study the classical limit of (5), we let $q = e^{-\eta}$ and consider the small η expansion of these equations. It was shown in [6] that when η is near 0, $R(x)$ can be expanded in the form (2). Now expanding (4) to the first order in η, we obtain

$$\begin{aligned} \Delta(\hat{e}_{\alpha_i}) &= \hat{e}_{\alpha_i} \otimes 1 + 1 \otimes \hat{e}_{\alpha_i} + \tfrac{1}{2}\eta[\hat{e}_{\alpha_i} \otimes h_{\alpha_i} - h_{\alpha_i} \otimes \hat{e}_{\alpha_i}] + O(\eta^2) \\ \Delta(\hat{f}_{\alpha_i}) &= \hat{f}_{\alpha_i} \otimes 1 + 1 \otimes \hat{f}_{\alpha_i} + \tfrac{1}{2}\eta[\hat{f}_{\alpha_i} \otimes h_{\alpha_i} - h_{\alpha_i} \otimes \hat{f}_{\alpha_i}] + O(\eta^2) \\ \Delta(h_{\alpha_i}) &= h_{\alpha_i} \otimes 1 + 1 \otimes h_{\alpha_i} \qquad \forall i. \end{aligned} \tag{8}$$

Inserting (8) into the first equation of (5), we arrive at

$$\begin{aligned} [\hat{e}_{\alpha_i} \otimes 1 + 1 \otimes \hat{e}_{\alpha_i}, r(x)] &= \hat{e}_{\alpha_i} \otimes h_{\alpha_i} - h_{\alpha_i} \otimes \hat{e}_{\alpha_i} + O(\eta) \\ [\hat{f}_{\alpha_i} \otimes 1 + 1 \otimes \hat{f}_{\alpha_i}, r(x)] &= \hat{f}_{\alpha_i} \otimes h_{\alpha_i} - h_{\alpha_i} \otimes \hat{f}_{\alpha_i} + O(\eta) \\ [h_{\alpha_i} \otimes 1 + 1 \otimes h_{\alpha_i}, r(x)] &= 0 \qquad \forall i \end{aligned} \tag{9}$$

where equation (2) has also been used. Since $\hat{e}_{\alpha_i}, \hat{f}_{\alpha_i}, h_{\alpha_i}$ reduce respectively to $e_{\alpha_i}, f_{\alpha_i} h_{\alpha_i}$, the simple and Cartan generators of the simple basic classical Lie superalgebra g, it follows from (9) that

$$\begin{aligned} [\partial(e_{\alpha_i}), r(x)] &= e_{\alpha_i} \wedge h_{\alpha_i} \\ [\partial(f_{\alpha_i}), r(x)] &= f_{\alpha_i} \wedge h_{\alpha_i} \\ [\partial(h_{\alpha_i}), r(x)] &= 0 \qquad \forall i \end{aligned} \tag{10a}$$

where the wedge product $\wedge$ is defined by

$$a \wedge b = (I \otimes I - T)(a \otimes b) \qquad \forall a, b \in g$$

with $I : g \to g$ the identity map on g.

In exactly the same way, we can obtain from the second equation of (5) the linear equation

$$[xe_{\alpha_0} \otimes 1 + 1 \otimes e_{\alpha_0}, r(x)] = xe_{\alpha_0} \otimes h_{\alpha_0} - h_{\alpha_0} \otimes e_{\alpha_0}. \tag{10b}$$

Equations (10) are the lowest order non-trivial terms in the η expansions of (5); thus their solutions must satisfy the quantum Yang–Baxter equation (1) up to the order η^2, that is, the classical Yang–Baxter equation [6]. We will give a more direct proof of this fact in the next section. Here we want to point out that (10) only gives rise to a subclass of trigonometric solutions of the graded classical Yang–Baxter equation, which may be quantized to obtain quantum R-matrices through quantizing the superalgebra g.

In order to derive a set of equations similar to (10) in the rational case, we recall that a rational R-matrix can be obtained from a given solution $R(x)$ of (5) by setting $x = q^u$, then taking the limit $q \to 1$,

$$R^{(0)}(u) = \lim_{q \to 1} R(q^u). \tag{11}$$

This limit always exists, and $R^{(0)}$ is a non-trivial rational function of u, which may be expanded in a power series in u^{-1},

$$R^{(0)}(u) = \chi(u)\left\{I \otimes I + \frac{r_1}{u} + \frac{r_2}{u^2} + \dots\right\} \tag{12}$$

with $\chi(u)$ a scalar function. It follows from the fact that $R^{(0)}(u)$ satisfies the quantum Yang–Baxter equation that

$$r(u)=\frac{r_1}{u} \tag{13}$$

solves the classical Yang-Baxter equation (3′).

In the limit $x=q^u$, $q\to 1$, the equations in (8) simply reduce to

$$[R^{(0)}(u), \partial(a)]=0 \qquad \forall a\in g. \tag{14}$$

Thus to the first order in the u^{-1} expansion, the above equation leads to

$$[r(u), \partial(a)]=0 \qquad \forall a\in g \qquad r(u)=r_1/u. \tag{15}$$

In the following section we will explicitly construct solutions of (10) and (15) in $g\otimes g$, and directly show that they indeed satisfy the graded classical Yang–Baxter equation. Such constructions are universal in the sense that they define valid solutions in all representations of g.

3. Classical r-matrices

In this section we construct the solutions (10) and (15) and show directly that they satisfy the graded classical Yang–Baxter equation. But before doing this, we establish some notation.

For g a simple basic classical Lie superalgebra, we denote the set of its positive roots by Φ^+. Then $\Phi^+=\Phi_0^+\cup\Phi_1^+$, where Φ_0^+ and Φ_1^+ are the sets of even and odd positive roots of g respectively. Choose the basis

$$g=\{e_\alpha, f_\alpha, h_\mu \mid \alpha\in\Phi^+, \mu=1,2,\ldots,N\} \tag{16}$$

such that under a fixed, nondegenerate, invariant bilinear form (,) on g, we have

$$(f_\alpha, e_\beta)=(-1)^{[\alpha][\beta]}(e_\beta, f_\alpha)=\delta_{\alpha,\beta} \qquad (h_\mu, h_\nu)=\eta_{\mu\nu}$$

where $\eta_{\mu\nu}$ is an arbitrarily chosen metric for the root space of g, $\det\eta\neq 0$, and

$$[\alpha]=\begin{cases}0 & \alpha\in\Phi_0^+\\ 1 & \alpha\in\Phi_1^+.\end{cases}$$

Now the graded commutation relations can be expressed as

$$\begin{aligned}
&[e_\alpha, e_\beta\}=N_{\alpha,\beta}e_{\alpha+\beta} && [f_\alpha, f_\beta\}=\bar N_{\alpha,\beta}f_{\alpha+\beta} &\\
&[e_\alpha, f_\beta\}=M_{\alpha,\beta}e_{\alpha-\beta} && \alpha>\beta &\\
&[e_\alpha, f_\beta\}=\bar M_{\alpha,\beta}f_{\beta-\alpha} && \alpha<\beta &\\
&[e_\alpha, f_\alpha\}=h_\alpha=\alpha(h^\mu)h_\mu && h^\mu=\eta^{\mu\nu}h_\nu & \eta^{\mu\nu}\eta_{\nu\rho}=\delta^\mu_\rho\\
&[h_\mu, e_\alpha]=\alpha(h_\mu)e_\alpha && [h_\mu, f_\alpha]=-\alpha(h_\mu)f_\alpha &\\
&[h_\mu, h_\nu]=0
\end{aligned} \tag{17}$$

with the structure constants $N_{\alpha,\beta}$, $\bar N_{\alpha,\beta}$ etc subject to the following constraints:

$$\begin{aligned}
N_{\alpha,\beta}&=-(-1)^{[\alpha](1+[\beta])}\bar M_{\alpha,\alpha+\beta}\\
\bar N_{\alpha,\beta}&=-(-1)^{[\beta](1+[\alpha])}M_{\alpha+\beta,\beta}\\
M_{\alpha,\beta}&=-(-1)^{[\alpha]+[\beta]+[\alpha][\beta]}M_{\alpha,\alpha-\beta}\\
\bar M_{\alpha,\beta}&=-(-1)^{[\alpha]+[\beta]+[\alpha][\beta]}\bar M_{\beta-\alpha,\beta}.
\end{aligned} \tag{18}$$

3.1. The solution of (15)

Consider (15) first, assuming that $r(x) \in g \otimes g$. The only element of $g \otimes g$ which commutes with $\partial(g)$ is, up to scalar multiples,

$$A = \tfrac{1}{2}[\partial(C) - C \otimes 1 - 1 \otimes C]$$

where C is quadratic Casimir of g. Thus r_1 must be proportional to A, and because (15) is invariant under scalar multiplication of $r(u)$, we may choose $r_1 = A$. Therefore

$$r(u) = A/u \tag{19}$$

and in the basis (16), A reads

$$A = \sum_{\mu=1}^{N} h^{\mu} \otimes h_{\mu} + \sum_{\alpha \in \Phi^{+}} \{f_{\alpha} \otimes e_{\alpha} + (-1)^{[\alpha]} e_{\alpha} \otimes f_{\alpha}\}. \tag{20}$$

It is straightforward to show that

$$[A_{12}, A_{13}] = -[A_{12}, A_{23}] = [A_{13}, A_{23}]$$

so that

$$\begin{aligned}[r_{12}(u), r_{13}(u+v)] + [r_{12}(u), r_{23}(v)] + [r_{13}(u+v), r_{23}(v)]& \\ = \left[\frac{1}{u(u+v)} - \frac{1}{uv} + \frac{1}{v(u+v)}\right][A_{12}, A_{13}] = 0.&\end{aligned} \tag{21}$$

Therefore, the $r(u)$ given in (19) satisfies the graded classical Yang–Baxter equation (3′).

3.2. The solution of (10)

Now we consider equation (10). Note that it implies

$$\begin{aligned} &[r(x) + r^{T}(y), \partial(a)] = 0 \qquad \forall a \in g \\ &[r(x) + r^{T}(x^{-1}), x e_{\alpha_0} \otimes 1 + 1 \otimes e_{\alpha_0}] = 0 \end{aligned} \tag{22}$$

where $r^{T}(x) = T(r(x))$. Equations (22) then lead to

$$[r(x) + r^{T}(x^{-1}), a \otimes 1] = [r(x) + r^{T}(x^{-1}), 1 \otimes a] = 0 \qquad \forall a \in g \tag{23}$$

which admits only the trivial solution in $g \otimes g$, i.e.

$$r(x) + r^{T}(x^{-1}) = 0 \tag{24}$$

and needless to say (24) satisfies the second of equations (22). Now differentiating the first of equations (22) with respect to x, we see that

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} r(x), \partial(a)\right] = 0 \qquad \forall a \in g \tag{25}$$

thus $\mathrm{d}r(x)/\mathrm{d}x$ is proportional to the operator $A \in g \otimes g$ defined in (20). Therefore, $r(x)$ can be expressed as

$$r(x) = r_0 + \rho(x) A \tag{26}$$

where $r_0 \in g \otimes g$ is x independent and $\rho(x)$ is a scalar function. It follows from (24) that

$$r_0 + T(r_0) = 0 \qquad \rho(x) = -\rho(x^{-1}). \tag{27}$$

In order to determine r_0 and $\rho(x)$, we need to go back to equations (10a) and (10b). The first of equations (27) requires that $r_0 \in g \wedge g$; since it also commutes with $\partial(h_\mu), \forall \mu$, it must be of the form

$$r_0 = \sum_{\alpha \in \Phi^+} \eta_\alpha (-1)^{[\alpha]} \{e_\alpha \otimes f_\alpha - (-1)^{[\alpha]} f_\alpha \otimes e_\alpha\} \tag{28}$$

where the η_α are scalar parameters which we now determine. Inserting (26) and (28) in (10a) we immediately see that

$$\eta_{\alpha_i} = 1 \qquad \text{for all simple roots } \alpha_i \qquad i = 1, 2, \ldots, N$$

$$\eta_{\alpha+\alpha_i} = \eta_\alpha \qquad \text{if } \alpha, \alpha+\alpha_i \in \Phi^+, \alpha_i \text{ is a simple root}$$

$$\eta_{\alpha-\alpha_i} = \eta_\alpha \qquad \text{if } \alpha, \alpha-\alpha_i \in \Phi^+, \alpha_i \text{ is a simple root}$$

and this uniquely determines

$$\eta_\alpha = 1 \qquad \forall \alpha \in \Phi^+.$$

Therefore

$$r_0 = \sum_{\alpha \in \Phi^+} (-1)^{[\alpha]} \{e_\alpha \otimes f_\alpha - (-1)^{[\alpha]} f_\alpha \otimes e_\alpha\}. \tag{29}$$

Finally, applying (26) and (28) to (10b) and utilizing the relations (18) amongst the structure constants of g, we can easily see that $\rho(x) = (1+x)/(1-x)$, so that

$$r(x) = r_0 + \frac{1+x}{1-x} A \tag{30}$$

with A and r_0 given by (20) and (29) respectively. Direct computation confirms that (30) does indeed satisfy (3).

4. Connection with Lie bi-superalgebras

In this section we briefly discuss the connection of equations (10) and (15) and their solutions with the theory of Lie bi-superalgebras.

Recall that a Lie superalgebra g is a $\mathbb{Z}_2$-graded vector space endowed with a bracket structure $[,\}: g \otimes g \to g$, which is (graded) skew-symmetric,

$$[,\} = -[,\} \cdot T$$

and satisfies the Jacobian identity

$$[,\} \cdot ([,\} \otimes I) - [,\} \cdot (I \otimes [,\}) = [,\} \cdot ([,\} \otimes I) \cdot (I \otimes T).$$

The concept of Lie superalgebras may be dualized, leading to that of Lie co-superalgebras. Let A be a $\mathbb{Z}_2$-graded vector space. It is a Lie co-superalgebra if there exists a co-bracket structure $\Delta_0: A \to A \otimes A$ which satisfies the following conditions:

(i) $\qquad \Delta_0 = -T \cdot \Delta_0$

(ii) $\qquad (\Delta_0 \otimes I) \cdot \Delta_0 - (I \otimes \Delta_0) \cdot \Delta_0 = (I \otimes T) \cdot (\Delta_0 \otimes I) \cdot \Delta_0 \qquad (31)$

(iii) $\qquad \Delta_0$ is parity preserving.

The second of these equations is known as the co-Jacobian identity. With regard to (iii), parity preserving means that if $\Delta_0(a)$, $a \in A$ is expressed as $\Delta_0(a) = \Sigma_s\, a_s^{(1)} \otimes a_s^{(2)}$ where $a_s^{(1)}$ and $a_s^{(2)}$ are homogeneous, then $[a] \equiv [a_s^{(1)}] + [a_s^{(2)}]$ (mod 2)$\forall s$. Because of (i), $\Delta_0(A) \subseteq A \wedge A$, where $A \wedge A$ is the vector space spanned by the elements $a \wedge b$, $a, b \in A$.

A Lie bi-superalgebra B is a Lie superalgebra with the bracket structure $[\,,\}: B \otimes B \to B$, and at the same time, a Lie co-superalgebra with the co-bracket $\Delta_0: B \to B \otimes B$ such that the following compatibility condition is satisfied:

$$\Delta_0([a, b\}) = [\Delta_0(a), \partial(b)\} + [\partial(a), \Delta_0(b)\} \qquad \forall a, b \in B. \tag{32}$$

For a given Lie superalgebra g, assume that there exists an even element r in $g \otimes g$ and define a map $\Delta_0: g \to g \otimes g$ by

$$\Delta_0(a) = [\partial(a), r] \qquad \forall a \in g \tag{33}$$

in analogy to equation ($10a$). Obviously, the Δ_0 defined this way satisfies (32) and also preserves parity. In order for Δ_0 to define a consistent co-bracket structure on g, it should also meet the first two requirements of (31); and this imposes rigid constraints on r. Multiplying both sides of (33) by T then adding the resultant equation to (33) itself we arrive at

$$(\Delta_0 + T \cdot \Delta_0)(a) = [\partial(a), r + r^T].$$

If $\Delta_0 + T \cdot \Delta_0 = 0$ as required by (i) of (31), then $0 = [\partial(a), r + r^T]$, i.e.

$$r + r^T = \rho A \qquad \rho \text{ a scalar.} \tag{34}$$

The co-Jacobian identity requires that

$$\begin{aligned} 0 &= [(\Delta_0 \otimes I) \cdot \Delta_0 - (I \otimes \Delta_0) \cdot \Delta_0 - (I \otimes T) \cdot (\Delta_0 \otimes I) \cdot \Delta_0](a) \\ &= [[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}], \partial^{(2)}(a)] \\ &\quad - [[\partial^{(2)}(a), r_{23} + r_{23}^T], r_{12}] \qquad \forall a \in g \end{aligned}$$

where

$$\partial^{(2)}(a) = a \otimes 1 \otimes 1 + 1 \otimes a \otimes 1 + 1 \otimes 1 \otimes a \qquad \forall a \in g.$$

Assume that (34) holds, then a sufficient condition to guarantee the co-Jacobian identity is that r satisfies the graded classical Yang–Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{35}$$

Therefore an r satisfying equations (34) and (35) defines a co-bracket structure Δ_0 on g through (33), which is compatible with the bracket structure of g, thus turning g into a Lie bi-superalgebra. Such a Lie bi-superalgebra is called quasitriangular.

Now we consider equation (10) and its solution (30). Set

$$r = r(0) = r_0 + A$$

where r_0 is given in (29). Then this r satisfies (35) and $r + r^T = 2A$. Therefore equation (10a) defines a co-bracket structure $\Delta_0: g \to g \otimes g$ on the simple basic classical Lie superalgebra g, which acting on the simple and Cartan generators of g gives

$$\begin{aligned} &\Delta_0(e_{\alpha_i}) = e_{\alpha_i} \wedge h_{\alpha_i} \\ &\Delta_0(f_{\alpha_i}) = f_{\alpha_i} \wedge h_{\alpha_i} \qquad \forall i = 1, 2, \ldots, N \\ &\Delta_0(h_\mu) = 0 \qquad \forall \mu = 1, 2, \ldots, N \end{aligned} \tag{36}$$

where α_i, $i=1,2,\ldots,N$, are the simple roots of g. Since the compatibility condition (32) is satisfied automatically in this case, Δ_0 is uniquely specified on the whole of g.

It is worthwhile to make the relationship between this co-bracket structure on g and the co-multiplication $\Delta: \mathrm{U}_q(g)\to \mathrm{U}_q(g)\otimes \mathrm{U}_q(g)$ more explicit at this stage. It can be easily shown for the simple and Cartan generators that

$$\Delta_0=\lim_{\eta\to 0}(\Delta-\bar{\Delta})/\eta \qquad q=e^{-\eta} \tag{37}$$

where $\bar{\Delta}=T\cdot\Delta$ as usual. To prove that (37) holds in general we need to show that (37) obeys the compatibility condition (32). For two homogeneous elements $\hat{a},\hat{b}\in \mathrm{U}_q(g)$ such that when $q\to 1$, $\hat{a}\to a$, $\hat{b}\to b$, with $a, b\in g$, we have

$$\begin{aligned}\Delta_0([a,b\})&=\lim_{\eta\to 0}\{\Delta(\hat{a}\hat{b})-\bar{\Delta}(\hat{a}\hat{b})-(-1)^{[\hat{a}][\hat{b}]}[\Delta(\hat{b}\hat{a})-\bar{\Delta}(\hat{b}\hat{a})]\}/\eta\\ &=\Delta_0(a)\partial(b)+\partial(a)\Delta_0(b)-(-1)^{[a][b]}[\Delta_0(b)\partial(a)+\partial(b)\Delta_0(a)]\\ &=[\Delta_0(a),\partial(b)\}+[\partial(a),\Delta_0(b)\}\end{aligned} \tag{38}$$

and this is nothing else but (32).

The full equations (10*a*), (10*b*) together with the spectral parameter-dependent $r(x)$ given in (20) may be regarded as defining a co-bracket structure on the loop algebra $\hat{g}=g\otimes\mathbb{C}[x,x^{-1}]$, which is a super Kac-Moody algebra without a central extension. More explicitly, $\hat{g}=\{a_m\,|\,a_m=ax^m, a\in g, m\in\mathbb{Z}\}$, and $\partial:\hat{g}\to\hat{g}\otimes\hat{g}$ is defined by

$$\partial(a_m)=a_m\otimes 1+1\otimes a'_m$$

where, on the right-hand side, $a_m=ax^m$, $a'_m=ay^m$. The simple and Cartan generators of $\hat{g}$ are $e_{\alpha_i}, f_{\alpha_i}, e_{\alpha_0}x, f_{\alpha_0}x^{-1}$ and h_μ. From the results of [6] we know that (10) implies

$$[x^{-1}f_{\alpha_0}\otimes 1+y^{-1}1\otimes f_{\alpha_0}, r(x/y)]=x^{-1}f_{\alpha_0}\otimes h_{\alpha_0}-h_{\alpha_0}\otimes y^{-1}f_{\alpha_0}. \tag{10c}$$

Therefore equations (10*a*, *b*, *c*) together define a co-bracket structure $\Delta_0:\hat{g}\to\hat{g}\otimes\hat{g}$ for the simple and Cartan generators of $\hat{g}$, which extends to all the elements of $\hat{g}$ through

$$\Delta_0(a_m)=[ax^m\otimes 1+1\otimes ay^m, r(x/y)] \qquad a\in g \qquad m\in\mathbb{Z} \tag{39}$$

and the fact that $r(x)$ given in (30) satisfies the graded classical Yang–Baxter equation ensures that Δ_0 satisfies the co-Jacobian identity.

Similarly the rational r-matrix given in (19) defines a co-bracket on $\hat{g}$ through

$$\Delta_0(a_m)=\left[ae^{mu}\otimes 1+1\otimes ae^{mv}, \frac{A}{u-v}\right]. \tag{40}$$

The problem of introducing a spectral parameter into the structure of a Lie bi-superalgebra will be examined in more detail elsewhere [15], together with a more systematic account of the theory of Lie bi-superalgebras.

5. Integrable systems

In this section we give a procedure for constructing integrable supersymmetric dynamical systems from given classical r-matrices for Lie superalgebras.

Consider a classical r-matrix $r(x) \in \pi(g) \otimes \pi(g)$, with π a finite-dimensional irreducible representation of the simple basic classical Lie superalgebra g. Let T_M, $M = 1, 2, \ldots, \dim g$, be a basis of g in this representation, satisfying the graded commutation relations

$$[T_M, T_N\} = f^P_{MN} T_P$$

with f^P_{MN} the structure constants of g. Then

$$r(x) = r^{MN}(x) T_M \otimes T_N \tag{41}$$

and because $r(x)$ is even, $r^{MN}(x) = 0 \quad \text{if} \quad [T_M] + [T_N] = 1.$

To construct integrable systems using the r-matrix, we assume that a supermanifold $\mathcal{M}$ with a non-degenerate symplectic metric ω is given. If X^A, $A = 1, 2, \ldots, \dim \mathcal{M}$, are the local coordinates of $\mathcal{M}$, we assign a Poisson bracket structure $\{,\}$ to $\mathcal{M}$ by requiring that for arbitrary functions $F(X)$, $G(X)$

$$\{F(X), G(X)\} = \omega^{AB}(X) \frac{\partial F(X)}{\partial X^B} \frac{\partial G(X)}{\partial X^A}. \tag{42}$$

Here ω^{BA} is the inverse of the metric ω, which is symplectic, i.e. $\omega^{AB} = -(-1)^{[A][B]}\omega^{BA}$, where $[A]$ is defined to be 0 if A is an even index, and 1 if A is odd, and similarly for $[B]$. A Poisson bracket structure obeys the Jacobian identity, thus we require ω to satisfy

$$(-1)^{[A][B]}\partial_A \omega_{BC} + (-1)^{[B][A]}\partial_B \omega_{AC} + (-1)^{[C][B]}\partial_C \omega_{AB} = 0. \tag{43}$$

Now we construct, from the X, dynamical variables $S_M = S_M(X)$, $M = 1, 2, \ldots, \dim g$, such that

$$\{S_M, S_N\} = f^P_{MN} S_P. \tag{44}$$

Then we define

$$Q(x) = S_M r^{MN}(x) T_N. \tag{45}$$

It follows directly from the fact that $r(x)$ satisfies the graded classical Yang–Baxter equation that

$$\{Q(x) \otimes Q(xy)\} + [Q(x) \otimes I + I \otimes Q(xy), r(y)] = 0 \tag{46}$$

where in the first term on the left-hand side of (46) we take the Poisson bracket among the S_M in the Q and the tensor product $\otimes$ for the T_N. Now multiply both sides of (46) by $n[Q(x) \otimes I]^{n-1}$ from the left, and $m[I \otimes Q(xy)]^{m-1}$ from the right with $m, n > 0$, then take the supertrace over the representation $\pi(g)$. The second term in (46) does not contribute anything, so we have

$$\begin{aligned} &\mathrm{Str}(n[Q(x) \otimes I]^{n-1}\{Q(x) \otimes Q(xy)\} m[I \otimes Q(xy)]^{m-1}) \\ &\qquad = \{\mathrm{Str}[Q^n(x)], \mathrm{Str}[Q^m(xy)]\} = 0 \qquad 0 < m, n \in \mathbb{Z}^+. \end{aligned} \tag{47}$$

Therefore, we may regard the coefficients of the Laurant expansions of

$$I_k(x) = \mathrm{Str}[Q^k(x)] \qquad 0 < k \in \mathbb{Z}^+ \tag{48}$$

as constants of motion of a certain dynamical system. If the number of the independent constants of motion is equal to half the dimension of the supermanifold, $\dim \mathcal{M}$, then this dynamical system is completely integrable.

It would be of interest to explicitly construct some integrable systems using this procedure; we hope to return to this elsewhere.

6. Conclusion

By examining the classical limit of the quantum supergroup equations developed in [6], we have constructed two sets of linear equations, the solutions of which automatically satisfy the graded classical Yang–Baxter equation. Their solutions have been obtained in explicit form and, although not covering all the classical r-matrices, they do constitute an important subclass of trigonometric and rational solutions of the graded classical Yang–Baxter equation, since they can be quantized to obtain quantum R-matrices, through the quantization of Lie superalgebras. As discussed in section 5, solutions of the graded classical Yang–Baxter equation may be applied to construct integrable supersymmetric dynamical systems.

Equations (10) and (15) reveal a deep connection between the theory of Lie bi-superalgebras and the graded classical Yang–Baxter equation. In fact, if we regard them as defining co-bracket structures on Lie superalgebras, then the fact that their solutions automatically satisfy the graded classical Yang–Baxter equation simply follows from the self-consistency of the co-bracket structures and their compatibility with the bracket structure of the Lie superalgebra. The theory of Lie bi-superalgebras will be developed more systematically in a separate paper [15], where the role of the graded classical Yang–Baxter equation in the theory of Lie bi-superalgebras is studied more closely.

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